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THE ARCHIMEDEAN GRAPH OF THE FREE SEMIGROUP*

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To my father Professor Putcha Venkateswarly

0. Introduction

Let X be a set, and $\mathcal{F}(X)$ the free semigroup on X. If $\omega_1, \omega_2 \in \mathcal{F}(X)$, define $\omega_1 \longrightarrow \omega_2$ to mean that ω_1 is a subword of a power of ω_2. Define $\omega_1 \;—\; \omega_2$ to mean $\omega_1 \longrightarrow \omega_2 \longrightarrow \omega_1$. We call $(\mathcal{F}(X), \longrightarrow)$ the directed archimedean graph and $(\mathcal{F}(X), —)$ the undirected archimedean graph of the free semigroup. The purpose of this paper is to study these graphs in detail.

1. Preliminaries

Throughout, S will denote a semigroup and $\mathbf{Z}^+$ the set of positive integers. We use the notation of [1], which we repeat for convenience.

1.1. Definition. Let S be a semigroup and $a, b \in S$.

(1) $a|b$ if $b \in S^1 a S^1$.

(2) $a \longrightarrow b$ if $a|b^i$ for some $i \in Z^+$.

(3) $a \;—\; b$ if $a \longrightarrow b \longrightarrow a$.

(4) A finite sequence $\langle x_i \rangle_{i=1}^n$ in S is said to be a *sequence from a to b* if

$$a \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \dots \longrightarrow x_n \longrightarrow b.$$

By $n = 0$ or $\langle x_i \rangle_{i=1}^n$ empty we mean $a \longrightarrow b$. $\langle x_i \rangle_{i=1}^n$ is *indecomposable* if $n > 0$ and no proper subsequence of $\langle x_i \rangle$ (including the empty sequence) is a sequence from a to b. $\langle x_i \rangle_{i=1}^n$ is *minimal* if $n > 0$ and no sequence of smaller length (including length 0) exists from a to b.

(5) A finite sequence $\langle x_i \rangle_{i=1}^n$ in S is said to be a *sequence between a and b* if

$$a \;—\; x_1 \;—\; x_2 \;—\; \dots \;—\; x_n \;—\; b.$$

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By $n = 0$ or $\langle x_i \rangle_{i=1}^n$ empty we mean $a — b$. $\langle x_i \rangle_{i=1}^n$ is *indecomposable* if $n > 0$ and no proper subsequence of $\langle x_i \rangle$ (including the empty sequence) is a sequence between a and b. $\langle x_i \rangle_{i=1}^n$ is *minimal* if $n > 0$ and no sequence of smaller length (including length 0) exists between a and b.

We call $(S, \longrightarrow)$ the *directed archimedean graph* and $(S, —)$ the *undirected archimedean graph* of S. If S is an $\mathcal{S}$-indecomposable semigroup, then according to [4], $(S, \longrightarrow)$ is connected, and according to [1] even $(S, —)$ is connected. Thus for each $a,b \in S$, S an $\mathcal{S}$-indecomposable semigroup, there will exist a sequence from a to b as well as a sequence between a and b. If $a \not\rightarrow b$, there will exist a minimal sequence from a to b, and if $a \not— b$, there will exist a minimal sequence between a and b.

1.2. Definition. Let S be a semigroup and $a,b \in S$. Then $d(a,b) = 0$ if $a — b$. If a, b lie in the same $\mathcal{S}$-indecomposable component but $a \not— b$, then $d(a,b)$ is the length of a minimal sequence between a and b. If a, b lie in different $\mathcal{S}$-indecomposable components of S, then $d(a, b) = \infty$.

1.3. Remark. For any $a,b,c \in S$,

$$d(a,c) \leqslant d(a,b) + d(b,c) + 1.$$

Let X be a set. Then $\mathcal{F} = \mathcal{F}(X)$ denotes the free semigroup on X, $|X|$ the cardinality of X. If $\omega \in \mathcal{F}$, then $|\omega|$ denotes the length of ω. Let $Y = \{A_1, ..., A_n\}$ be a finite subset of X. Then the free content

$$\mathcal{C} = \mathcal{C}(Y) = \mathcal{C}(A_1, ..., A_n)$$

on Y is the subsemigroup of $\mathcal{F}$ consisting of all words involving exactly the letters $A_1, ..., A_n$. According to [3], the $\mathcal{S}$-indecomposable components of $\mathcal{F}$ are just $\mathcal{C}(Y)$, Y a finite subset of X. Let $\omega_1, \omega_2 \in \mathcal{C}$. The purpose of this paper is threefold: firstly we produce small sequences between ω_1 and ω_2, secondly we show the number of minimal sequences between ω_1 and ω_2 to be finite, and finally we obtain some general information about the nature of $(\mathcal{C}, —)$.

2. The case of two letters

Let $\mathcal{F} = \mathcal{F}(A,B)$, $\mathcal{C} = \mathcal{C}(A,B)$, and let $\omega_1, \omega_2 \in \mathcal{C}$. We want to find small sequences between ω_1 and ω_2. First let $\omega \in \mathcal{C}$ and consider the following property.

(α) For some $U \in \mathcal{F}$, $|U| \geqslant 2$, either $U^2 | \omega$ or $\omega = ST(S, T \in \mathcal{F}^1)$ and $TS = UJU$ for some $J \in \mathcal{F}^1$.

2.1. Lemma. *Let $\omega \in \mathcal{C}(A,B)$, and suppose ω does not start and end with the same letter. Then ω does not satisfy (α) if and only if*

$$\omega \in K = \{A^iB^j \mid 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 3\} \cup \{B^iA^j \mid 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 3\}.$$

Proof. We may assume that ω starts with A and ends with B. So

$$\omega = A^{\epsilon_1}B^{\delta_1} \dots A^{\epsilon_n}B^{\delta_n},$$

where $\epsilon_i, \delta_i \geqslant 1$ $(i = 1, \dots, n)$. Assume that ω does not satisfy (α). We are going to show that $n = 1$. For suppose $n > 1$. Choose j so that ϵ_j is minimal. Assume first that $1 < j < n$. Let

$$V = A^{\epsilon_{j-1}}B^{\delta_{j-1}}A^{\epsilon_j}B^{\delta_j}A^{\epsilon_{j+1}}.$$

Then $\epsilon_{j-1} \geqslant \epsilon_j$ and $\epsilon_{j+1} \geqslant \epsilon_j$. So $(A^{\epsilon_j}B^{\delta_{j-1}})^2 | V$ or $(B^{\delta_j}A^{\epsilon_j})^2 | V$ depending on whether $\delta_{j-1} \geqslant \delta_j$ or $\delta_j \geqslant \delta_{j-1}$. As $V | \omega$, this leads to a contradiction. Assume next that $j = 1$. Either $\delta_1 \geqslant \delta_n$, in which case $\omega = UJU$, where $U = A^{\epsilon_1}B^{\delta_n}$, or else $\delta_n > \delta_1$ and $\omega = ST$ with $T = B^{\delta_1} \dots A^{\epsilon_n}B^{\delta_n}$, $S = A^{\epsilon_1}$, and $TS = UJU$, where $U = B^{\delta_1}A^{\epsilon_1}$. Thus ω satisfies (α), a contradiction. Assume now the remaining case that $j = n$. Either $\delta_{n-1} \leqslant \delta_n$ in which case $(A^{\epsilon_n}B^{\delta_{n-1}})^2 | \omega$, or else $\omega = ST$ with $S = A^{\epsilon_1}B^{\delta_1} \dots A^{\epsilon_n}$, $T = B^{\delta_n}$, and $TS = UJU$, where $U = B^{\delta_n}A^{\epsilon_n}$. Again ω satisfies (α), a contradiction. So $n = 1$, and then evidently $\epsilon_1 \leqslant 3$ and $\delta_1 \leqslant 3$.

2.2. Theorem. *Let $\omega_1, \omega_2 \in \mathcal{C}(A,B)$. Then there exists a minimal sequence between ω_1 and ω_2 of length at most $|\omega_1| + |\omega_2| - 3$.*

Proof. Let

$$Q = \{(\omega_1, \omega_2) \mid \omega_1, \omega_2 \in \mathcal{C},\ d(\omega_1, \omega_2) \leqslant |\omega_1| + |\omega_2| - 3\}.$$

In $\mathbf{Z}^+ \times \mathbf{Z}^+$, set $(i,j) \leqslant (k,l)$ iff $i \leqslant k$ and $j \leqslant l$. Let

$$P = \{(i,j) \mid (\omega_1, \omega_2) \in Q \text{ whenever } |\omega_1| \leqslant i \text{ and } |\omega_2| \leqslant j\}.$$

We assume $P \neq \mathbf{Z}^+ \times \mathbf{Z}^+$ and get a contradiction. Find $(m,n) \notin P$ minimal with respect to $\leqslant$. There exists $(\omega_1, \omega_2) \notin Q$ with $|\omega_1| = m$ and $|\omega_2| = n$. We claim that ω_1 does not satisfy (α). First suppose $\omega_1 = XU^2Y$, $|U| \geqslant 2$. Then

$$\omega_1 \text{ --- } UYXY \text{ --- } UYX = \omega_1', \qquad |\omega_1'| \leqslant m - 2.$$

So $d(\omega_1', \omega_2) \leqslant m + n - 5$. Thus

$$d(\omega_1, \omega_2) \leqslant m + n - 3,$$

a contradiction. Assume next that $\omega_1 = ST$, $TS = UJU$. Then

$$\omega_1 \text{ --- } TS \text{ --- } UJ = \omega_1', \qquad |\omega_1'| \leqslant m - 2,$$

and we get a contradiction as above. Also ω_1 cannot start and end with the same letter. For if say $\omega_1 = AXA$, then

$$\omega_1 \text{ --- } AX = \omega_1', \qquad |\omega_1'| = m - 1.$$

So

$$\begin{aligned} d(\omega_1, \omega_2) &\leqslant d(\omega_1, \omega_1') + d(\omega_1', \omega_2) + 1 = 1 + d(\omega_1', \omega_2) \\ &\leqslant m + n - 3, \end{aligned}$$

a contradiction. By Lemma 2.1, $\omega_1 \in K$. Similarly $\omega_2 \in K$. But it is easy to see that $K \times K \subseteq Q$. For example,

$$AB \text{ --- } BAB \text{ --- } B^2AB \text{ --- } AB^3 \text{ --- } AB^3A \text{ --- } A^2B^3A \text{ --- } B^3A^3,$$

$$AB^3 \text{ --- } B^2AB \text{ --- } BAB \text{ --- } AB \text{ --- } ABA \text{ --- } ABA^2 \text{ --- } A^3B$$

This proves the theorem. □

3. More than two letters

First let us note the following sequence in any semigroup S and $x, y, z, \omega \in S^1$:

(1) $$\begin{aligned} &xyz\omega \text{ --- } z\omega xyz\omega x \text{ --- } yz\omega xz\omega xyz\omega xz \text{ --- } z\omega xyz\omega xzyz\omega x \\ &\text{ --- } yz\omega xzyz\omega x \text{ --- } xzyz\omega \text{ --- } \omega xzyz\omega xzy \text{ --- } \omega xzyz\omega xzy\omega xz \\ &\text{ --- } z\omega xzy\omega xz\omega xzy \text{ --- } \omega xzy\omega xz \text{ --- } xzy\omega. \end{aligned}$$

In particular we have

(2) $$d(xyz\omega, xzyz\omega) \leqslant 4,$$

(3) $$d(xzy\omega, xzyz\omega) \leqslant 4,$$

(4) $$d(xyz\omega, xzy\omega) \leqslant 9.$$

In (1), substituting $\omega = 1$ we get a sequence between xyz and xzy of length 9. This is evidently also a sequence between xyz and zyx. Thus

(5) $$d(xyz, zyx) \leqslant 9.$$

Let us note that in $\mathcal{C}(A,B,C)$, the following is a minimal sequence:

$$ABC \longrightarrow BCBA \longrightarrow CBACB \longrightarrow ACB.$$

3.1. Problem. In $\mathcal{C}(A,B,C)$, (1) yields a sequence between ABC and ACB of length 9. Is this minimal? We can show that $d(ABC,ACB) \geqslant 6$. Working with (1) we can produce 12 sequences between ABC and ACB of length 9. They can be divided into 3 groups, each containing 4 sequences all of which differ only at the central word. The conjecture is that there are no more sequences of length 9 between ABC and ACB. The number of sequences is very much a function of the semigroup involved. By introducing the relations $A^2 = A$, $B^2 = B$, $C^2 = C$, we can produce $4 \cdot 3^{10}$ sequences of length 9 between ABC and ACB. We also conjecture that $d(ABC,ACB) = 9$ in this new semigroup, which of course would prove it for $\mathcal{C}(A, B, C)$.

3.2. Remark. That the sequence between ABC and ACB given in [1] can be much shortened was noticed independently by Professor B.M. Schein.

3.3. Theorem. *Let* $\omega_1, \omega_2 \in \mathcal{C}(A, \ldots, A_k)$, $k \geqslant 3$. *Then there exists a sequence between* ω_1 *and* ω_2 *of length less than* $5(|\omega_1| + |\omega_2|) - 20$ *involving words of length at most* $3M$, *where* $M = \max\{|\omega_1|, |\omega_2|\}$.

Proof. By using (1), (2) and (3), we can drop the extra letters and change ω_1 to a permutation U in $5(|\omega_1| - k)$ steps. A careful examination also shows that the words used are of length less than $3|\omega_1| \leqslant 3M$. Similarly we change ω_2 to a permutation V in $5(|\omega_2| - k)$ steps and using only words of length less than $3|\omega_2|$. Without loss of generality we may assume that $U = A_1 \ldots A_k$ and $V = A_{\sigma(1)} \ldots A_{\sigma(k)}$. First assume that $\sigma(i) < \sigma(j)$ for some $i < j$. Then fixing $A_{\sigma(i)}$ and $A_{\sigma(j)}$ in V, move the rest of the $k-2$ A's at most once so as to obtain U. This can be done in at most $10(k-2)$ steps by (1) and (4). If in (1) we always put z for the letter being moved, it is clear that we use only words of length at most $3K \leqslant 3M$. Assume next the other possibility that for all $i < j$, $\sigma(i) > \sigma(j)$. Then $V = A_k \ldots A_1$. By (1) and (5) we change V to $V_1 = A_1 A_2 A_k \ldots A_3$ in 10 steps using words of length at most $3K$. Now fixing A_1, A_{k-1} and A_k in V_1, move the $k-3$ remaining letters at most once (always substituting z in (1) for the letter being moved) so as to obtain U in at most $10(k-3)$ steps and using only words of length at most $3k$. Thus in either case U is changed to V in at most $10(k-2)$ steps involving words of length at most $3k$. So ω_1 is changed to ω_2 in at most

$$5(|\omega_1| - k) + 5(|\omega_2| - k) + 10(k-2) = 5(|\omega_1| + |\omega_2| - 4)$$

steps and involving words of length at most $3M$. So the sequence between ω_1 and ω_2 has length less than $5(|\omega_1| + |\omega_2| - 4)$. □

3.4. Remark. (i) Originally we had $10M - 20$ as the bound for the length of the sequence. We are grateful to Professor John Shafer for suggesting this improvement.

(ii) John Shafer has pointed out that a longer sequence can be obtained in a nicer way: Using (1) and (3), keep adding letters to ω_1 so as to change ω_1 to $\omega_1\omega_2$ in $5|\omega_2|$ steps. Then using (1) and (2), drop the letters on the left so as to obtain ω_2 in $5|\omega_1|$ steps. This yields a sequence between ω_1 and ω_2 of length less than $5|\omega_1| + |\omega_2|)$. However, the words used are longer than above.

3.5. Problem. Let $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_1 \not\rightarrow \omega_2$. For a sequence $\langle U_i \rangle_{i=1}^n$ between ω_1 and ω_2, let $r(\langle U_i \rangle) = \max_i |U_i|$. Let $\langle U_i \rangle$ be a sequence between ω_1 and ω_2. Does there necessarily exist a minimal sequence $\langle V_j \rangle$ between ω_1 and ω_2 such that $r(\langle V_j \rangle) \leqslant r(\langle U_i \rangle)$?

3.6. Problem. In Theorem 3.3, if k is left as a variable, it seems probable (looking at $A_1 \ldots A_k$ and $A_k \ldots A_1$) that the bounds given are best possible. Improve the bounds by using k in the formula as Section 2 does for $k = 2$.

Using John Shafer's idea (Remark 3.4(ii)) we prove:

3.7. Theorem. *Let $\omega_1, \omega_2 \in \mathcal{C}(A_1, \ldots, A_k)$. Then there exists a sequence from ω_1 to ω_2 of length less than* $2|\omega_2|$.

Proof. In any semigroup S with $x, y, z \in S^1$, we have

$$xzyz \longrightarrow zyxzy \longrightarrow xzy.$$

So by dropping letters from $\omega_1\omega_2$ on the right we obtain a sequence from $\omega_1\omega_2$ to ω_2 of length less than $2|\omega_2|$. This evidently is also a sequence from ω_1 to ω_2. □

In Theorem 3.3, the sequences are unnecessarily long if ω_1 and ω_2 differ only slightly. As such we have the following theorem which is proved in a similar manner as Theorem 3.3.

3.8. Theorem. *Let X be a set and Y a finite subset of X. Let $U, V \in \mathcal{F}(X)^1$ and $\omega_1, \omega_2 \in \mathcal{C}(Y)$. Then there exists a sequence between $U\omega_1 V$ and $U\omega_2 V$ of length less than* $5(|\omega_1| + |\omega_2|) - 10$.

4. Minimal and indecomposable sequences

Let $\omega_1, \omega_2 \in \mathcal{C} = \mathcal{C}(A_1, ..., A_k)$, $\omega_1 \not- \omega_2$. Then from any sequence between ω_1 and ω_2 we can extract an indecomposable sequence between ω_1 and ω_2.

4.1. Lemma. *Let* $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_2 \longrightarrow \omega_1$, $|\omega_2| \geq (K+1)|\omega_1| - 1$. *Then* $\omega_1^K | \omega_2$.

Proof. We have $\omega_2 = U\omega_1^j V$, U and V possibly empty, $|U|, |V| < |\omega_1|$. Thus

$$(K+1)|\omega_1| - 1 \leq |\omega_2| \leq j\,|\omega_1| + 2(|\omega_1| - 1),$$

from which we get $K < j + 1$. □

4.2. Lemma. *Let* $U, V \in \mathcal{C}$, $U^{2^{k+1}} \longrightarrow V$, $|V| \geq (2^k + 1)|U|$. *Then* $U^{2^k} | V$.

Proof. $U^{2^{k+1}}$ is a subword of V^i, i minimal. If $i = 1$, there is nothing to prove. So assume $i > 1$. We can write $V = \gamma U^s \alpha$, $V^{i-1} = \beta U^t A$, $\alpha\beta = U$, $s + t + 1 = 2^{k+1}$. So $s > t$ or $t > s$. If $s > t$, then $2^k \leq s$ and we are done. If $t > s$, then $2^k \leq t$. Then as $|\beta| \leq |U|$ and $|V| \geq 2^k|U| + |U|$, the first V in V^{i-1} must contain on the left at least βU^{2^k}. □

4.3. Lemma. *Let* $\langle U_i \rangle_{i=1}^n$ *be an indecomposable sequence between* ω_1 *and* ω_2. *Then it cannot be that for each* $i = 1, ..., n$,

$$|U_i| \geq (2^{n-1} + 1)M - 1,$$

where $M = \max\{|\omega_1|, |\omega_2|\}$.

Proof. Suppose

$$|U_i| \geq (2^{n-1} + 1)M - 1, \quad i = 1, ..., n.$$

Then by Lemma 4.1, $\omega_1^{2^{n-1}} | U_1$. So $\omega_1^{2^{n-1}} \longrightarrow U_2$. By Lemma 4.2, $\omega_1^{2^{n-2}} | U_2$. Continuing this way we get $\omega_1 | U_n$. So $\omega_1 \longrightarrow \omega_2$. Similarly $\omega_2 \longrightarrow \omega_1$, whence $\omega_1 — \omega_2$, a contradiction. □

4.4. Theorem. *Let* $\omega_1, \omega_2 \in \mathcal{C}$. *Then any indecomposable sequence between* ω_1 *and* ω_2 *of length* $\leq n$ *involves only words of length at most* $2^{(n^2+n)/2}(M-1)$, *where* $M = \max\{|\omega_1|, |\omega_2|\}$.

Proof. For $n = 1$, Lemma 4.3 yields the result. So we assume $n > 1$ and prove the result by induction on n. Let $\langle U_i \rangle_{i=1}^n$ be an indecomposable sequence between ω_1 and ω_2. By Lemma 4.3,

$$|U_i| \leqslant (2^{n-1}+1)M-2$$

for some i. Then $\langle U_j\rangle_{j=1}^{i-1}$ (unless $j=1$) and $\langle U_j\rangle_{j=i+1}^{n}$ (unless $j=n$) are indecomposable sequences between ω_1 and U_i, and U_i and ω_2, respectively. The result now follows by induction. □

Let $\omega_1,\omega_2 \in \mathcal{C}$, $\omega_1 \not\!\!- \omega_2$. Theorem 3.1 gives us a concrete sequence between ω_1 and ω_2 from which we can extract an indecomposable sequence between ω_1 and ω_2. Then Theorem 4.4 tells us that we need only scan through a finite number of words to find a minimal sequence (in fact all minimal sequences) between ω_1 and ω_2. However, the number of words to be scanned through fast becomes astronomically large. The question of improving Theorem 4.4 remains open.

4.5. Conjecture. The bound in Theorem 4.4 can be improved to $2^n(M-1)$. As in combinatorics, obtaining good bounds seems hard.

4.6. Corollary. *Let $\omega_1,\omega_2 \in \mathcal{C}$. Then the number of indecomposable sequences between ω_1 and ω_2 of length $\leqslant n$ is finite.*

The above corollary is not true if "indecomposable" is removed. For example in $\mathcal{C}(A,B)$,

$$ABAB — (AB)^n — AB — ABA, \quad n=2,3,\ldots, \qquad ABAB \not\!\!- ABA,$$

and for each n the points in the sequence are distinct. But this produces an infinite number of sequences of length 2 between $(AB)^2$ and ABA, viz. $\langle (AB)^n, AB\rangle$, $n=2,3,\ldots$.

4.7. Corollary. *Let $\omega_1,\omega_2 \in \mathcal{C}$. Then the number of minimal sequences between ω_1 and ω_2 is finite.*

In contrast we have in $\mathcal{C}(A,B)$,

$$A^2B \longrightarrow (AB)^nA \longrightarrow AB \quad \text{for all } n,$$

whence the number of minimal sequences from ω_1 to ω_2 can be infinite.

4.8. Theorem. *Let $\omega_1,\omega_2 \in \mathcal{C}$, $\omega_1 \not\!\!- \omega_2$. Let $M \in \mathbf{Z}^+$. Then there exists an indecomposable sequence $\langle V_i\rangle$ between ω_1 and ω_2 such that for each i, $\omega_1^M | V_i$ or $\omega_2^M | V_i$. In particular there exists an indecomposable sequence involving only words of length $\geqslant M$.*

Proof. Let

$$I = \{U \mid U \in \mathcal{C},\ \omega_1^m | U \text{ or } \omega_2^M | U\}.$$

Then $\omega_1^M, \omega_2^M \in I$, $\omega_1^M \not\longrightarrow \omega_2^M$. I is an ideal of $\mathcal{C}$ and hence by [5], I is $\mathcal{S}$-indecomposable. By [1] there exists a (non-empty) sequence $\langle U_i \rangle$ in I between ω_1^M and ω_2^M. But then $\langle U_i \rangle$ is a sequence between ω_1 and ω_2. Since $\omega_1 \not\longrightarrow \omega_2$, we can extract an indecomposable subsequence of $\langle U_i \rangle$ between ω_1 and ω_2. □

Combining Theorem 4.8 with Theorem 4.4, we have:

4.9. Corollary. *Let $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_1 \not\longrightarrow \omega_2$, and let $N \in \mathbf{Z}^+$. Then there exists an indecomposable sequence between ω_1 and ω_2 of length $\geq N$.*

4.10. Conjecture. Let $\omega_1, \omega_2 \in \mathcal{C}$. Suppose there exists an indecomposable sequence between ω_1 and ω_2 of length n. Then there exists an indecomposable sequence between ω_1 and ω_2 of length $n + 1$.

At this point we do not have any justification for the above conjecture, except that it is true with $\omega_1 = AB$ and $\omega_2 = A^2B$, viz.

$$\langle (AB)^{2^n} ABA,\ (AB)^{2^{n-1}} ABA (AB)^{2^{n-1}}, \ldots, AB(ABA)AB,\ ABAAB \rangle$$

is an indecomposable sequence between AB and A^2B of length $n + 2$ ($n \geq 0$), while $\langle ABA \rangle$ is a sequence of length 1.

5. The nature of the archimedean graph

The author [1,2] obtained some relationships existing among minimal sequences between any two points in certain large classes of semigroup. We shall now show that a similar result does not hold in the free semigroup.

5.1. Definition. By an *irreducible n-gon* ($n \geq 3$) in ($\mathcal{C}$, —) we mean a set of n disctinct points $u_1, \ldots, u_n$ such that

$$u_1 — u_2 — \ldots — u_n — u_1$$

and no other —'s except the trivial ones (u_1 — u_1, etc.) can be drawn.

5.2. Lemma. *Let $\omega_1, \omega_2 \in \mathcal{C}(A, B)$, $\omega_1 \longrightarrow \omega_2$. Suppose $A^{k+1} \nmid \omega_2$. Then $A^{2k+1} \nmid \omega_1$.*

Proof. This follows by noting that $A^{2k+1} \nmid \omega_2^i$ for any i. □

5.3. Corollary. *Let $\omega_1, \omega_2 \in \mathcal{C}(A,B)$, and let $\langle U_i \rangle_{i=1}^{t}$ be a sequence from ω_1 to ω_2. If $A^{k+1} \nmid \omega_2$, then $A^{2^{t+1}k+1} \nmid \omega_1$.*

We now produce irreducible n-gons for every $n \geqslant 3$ in $\mathcal{C} = \mathcal{C}(A,B)$. First we have

$$\begin{tikzcd}[column sep=small]
 & ABA \arrow[dr, dash] & \\
AB \arrow[ur, dash] \arrow[dr, dash] & & (ABA)AB \\
 & A(BAB)A \arrow[ur, dash] &
\end{tikzcd}$$

For $n \geqslant 1$,

$$\begin{tikzcd}[column sep=small]
 & ABA \arrow[r, dash] & A^2BA^2 \arrow[r, dash] & \cdots \arrow[r, dash] & A^{2^{n-1}}BA^{2^{n-1}} \arrow[r, dash] & ABA^{2^n} \arrow[dr, dash] & \\
AB \arrow[ur, dash] \arrow[dr, dash] & & & & & & ABA^{2^n+1}B. \\
 & A(BAB)A \arrow[r, dash] & A^2(BAB)A^2 \arrow[r, dash] & \cdots \arrow[r, dash] & A^{2^{n-1}}(BAB)A^{2^{n-1}} \arrow[r, dash] & A(BAB)A^{2^n} \arrow[ur, dash] &
\end{tikzcd}$$

Thus we have produced irreducible n-gons, $n \geqslant 4$, n even. Also the above yield, for $n \geqslant 0$, two sequences between AB and $ABA^{2^n+1}B$ $(n \geqslant 0)$ of length $n+1$. The sequences are minimal. For let $\langle U_i \rangle_{i=1}^{t+1}$ be a sequence from $ABA^{2^n+1}B$ to AB. Evidently $A^2 \nmid U_{t+1}$. By Corollary 5.3, $A^{2^{t+1}+1} \nmid \omega$. So $t \geqslant n$. This shows that in fact these sequences are minimal sequences *from* $ABA^{2^n+1}B$ to AB. We also have

$$\begin{tikzcd}[column sep=small]
 & (AB)^2 \arrow[dr, dash] & \\
AB \arrow[ur, dash] \arrow[rr, dash] & & (AB)^3
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
AB \arrow[r, dash] \arrow[d, dash] & ABABABA \arrow[dr, dash] & \\
ABA \arrow[r, dash] & ABAAB \arrow[r, dash] & ABABAAB
\end{tikzcd}$$

and for $n \geqslant 1$,

$$\begin{tikzcd}[column sep=small]
 & ABA \arrow[r, dash] & A^2BA^2 \arrow[r, dash] & \cdots \arrow[r, dash] & A^{2^{n-1}}BA^{2^{n-1}} \arrow[r, dash] & BA^{2^n} \arrow[r, dash] & BA^{2^n}B \arrow[d, dash] \\
AB \arrow[ur, dash] \arrow[dr, dash] & & & & & & \\
 & BAB \arrow[r, dash] & AB^2 \arrow[r, dash] & AB^2A \arrow[r, dash] & A^2B^2A^2 \arrow[r, dash] & \cdots \arrow[r, dash] & A^{2^{n-1}}B^2A^{2^{n-1}}.
\end{tikzcd}$$

We have thus produced irreducible n-gons in $\mathcal{C}(A,B)$ for every $n \geqslant 3$.

5.4. Theorem. *Let $\mathcal{C} = \mathcal{C}(A_1, \ldots, A_k)$, $k \geqslant 2$, $\omega \in \mathcal{C}$. Then for every $n \geqslant 3$, ω is a vertex of infinitely many irreducible n-gons.*

Proof. Let j be the largest positive integer such that $\omega = UV$ with $U, V \in \mathcal{F}^1$ and $VU = A_1^j C$. Then C does not start or end with A_1 and $A_1^{j+1} \nmid C$. Now let $i, n \in \mathbf{Z}^+$, $i \geqslant 2$, $n \geqslant 3$. In the irreducible n-gon given previously in $\mathcal{C}(A, B)$, replace AB by ω, and at every other vertex replace A by A_1^j and B by $(CA_1^j)^i C$. By the way we have chosen j, the polygon will remain irreducible in $\mathcal{C}(A_1, \ldots, A_k)$. Letting i vary, we have infinitely many irreducible n-gons each of which has ω as a vertex. □

5.5. Theorem. *Let $\omega_1, \omega_2 \in \mathcal{C} = \mathcal{C}(A_1, \ldots, A_k)$, $\omega_1 \neq \omega_2$. Let $n > 3$. Then the number of irreducible n-gons having both ω_1 and ω_2 as vertices is finite.*

Proof. First suppose $\omega_1 \not\!\!— \omega_2$. Then every irreducible n-gon containing ω_1 and ω_2 breaks up into 2 indecomposable sequences between ω_1 and ω_2 of length $< n$, and the result follows by Theorem 4.4.

So assume $\omega_1 — \omega_2$. We assume that there exist an infinite number of irreducible polygons containing ω_1 and ω_2. These can be described as sequences of words $\langle U_j^{(i)} \rangle_{j=1}^{n-2}$, $i \in \mathbf{Z}^+$, such that

$$\omega_1 — U_1^{(i)} — \ldots — U_{n-2}^{(i)} — \omega_2 .$$

If for any j, $\langle |U_j^{(i)}| \rangle_{i=1}^{\infty}$ is a bounded sequence, then we would have a contradiction to Theorem 4.4. So for every $M \in \mathbf{Z}^+$ there exists $i \in \mathbf{Z}^+$ such that $|U_j^{(i)}| > M$, $j = 1, \ldots, n-2$. Let $m \in \mathbf{Z}^+$. Then for M large enough and a corresponding i we can use Lemmas 4.1 and 4.2 continuously so as to obtain that $\omega_1^m | U_{n-2}^{(i)}$. So $\omega_1^m \longrightarrow \omega_2$ for any $m \in \mathbf{Z}^+$. Similarly $\omega_2^m \longrightarrow \omega_1$ for any $m \in \mathbf{Z}^+$. From the above, choose an i such that $\omega_1 | U_{n-2}^{(i)}$. Since $U_{n-2}^{(i)} \longrightarrow \omega_2$, there exists $m \in \mathbf{Z}^+$ such that $U_{n-2}^{(i)} | \omega_2^m$. Since $\omega_2^m \longrightarrow \omega_1$ we have $U_{n-2}^{(i)} \longrightarrow \omega_1$. So $U_{n-1}^{(i)} — \omega_1$ contradicting the irreducibility of the polygon. □

5.6. Definition. Let $\omega_1, \omega_2 \in \mathcal{C} = \mathcal{C}(A_1, \ldots, A_k)$. Then $\omega_1 \rightsquigarrow \omega_2$ if $\omega_1^i \longrightarrow \omega_2$ for every $i \in \mathbf{Z}^+$.

5.7. Lemma. *Let $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_1 \rightsquigarrow \omega_2$. Then $\omega_2 \rightsquigarrow \omega_1$, and in particular $\omega_1 — \omega_2$.*

Proof. Let $j \in \mathbf{Z}^+$. Set $i = (j+1)|\omega_2|$. Then $\omega_1^i \longrightarrow \omega_2$. By Lemma 4.1, $\omega_2^j | \omega_1^i$. So $\omega_2^j \longrightarrow \omega_1$ for any $j \in \mathbf{Z}^+$. □

5.8. Problem. Analyze the structural meaning of $\omega_1 \rightsquigarrow \omega_2$.

In contrast to Theorem 5.5, we now prove:

5.9. Theorem. *Let $\omega_1, \omega_2 \in \mathcal{C} = \mathcal{C}(A_1, \ldots, A_t)$, $t \geq 2$, $\omega_1 \neq \omega_2$. If ω_1 and ω_2 are vertices of an irreducible n-gon, $n > 4$, then there exists an infinite number of irreducible polygons, each with more than three vertices and each containing both ω_1 and ω_2.*

Proof. First assume that $\omega_1 \text{---} \omega_2$. The assumption implies that there exist $U, V \in \mathcal{C}$ such that $\omega_1 \text{---} U$, $\omega_2 \text{---} V$, $U \not\text{---} V$, $U \not\text{---} \omega_2$ and $V \not\text{---} \omega_1$. By Lemma 5.7, there exists $M \in \mathbf{Z}^+$ such that $U^M \not\twoheadrightarrow \omega_2$ and $V^M \not\twoheadrightarrow \omega_1$. Let $k \geq M$. By Theorem 4.8, there exists an indecomposable sequence $\langle D_i \rangle_{i=1}^m$ between U and V such that for each i either $U^k | D_i$ or $V^k | D_i$. Set $D_0 = U$ and $D_{m+1} = V$. Let p be the largest integer such that $0 \leq p \leq m$ and $\omega_1 \text{---} D_p$. If $p > 0$, then since $k \geq M$, we have that $V^k \nmid D_p$ and so $U^k | D_p$. Thus in any case $D_p \not\text{---} \omega_2$. Next let q be the smallest integer such that $p < q \leq m+1$ and $D_q \text{---} V$. As above, $D_q \not\text{---} U$. So we have the irreducible polygon,

$$\omega_1 \text{---} D_p \text{---} D_{p+1} \text{---} \ldots \text{---} D_q \text{---} \omega_2,$$

with $q - p \geq 1$ and at least one vertex of length $> k$. Since k is any integer $\geq M$, we have produced an infinite number of irreducible polygons each with more than three vertices and containing ω_1 and ω_2.

Next assume $\omega_1 \not\text{---} \omega_2$. By Theorem 4.8, there exists an indecomposable sequence $\langle U_i \rangle_{i=1}^m$ between ω_1 and ω_2 such that

$$|U_i| \geq 2(|\omega_1| + |\omega_2|)$$

for each $i = 1, \ldots, m$. Set $U_0 = \omega_1$ and $U_{m+1} = \omega_2$. By Lemma 4.1, $\omega_1 | U_1$. So $\omega_1 \longrightarrow U_2$. If $U_1 \rightsquigarrow \omega_1$, then we would have $U_2 \longrightarrow \omega_1$, whereupon $U_2 \text{---} \omega_1$, a contradiction. So $U_1 \not\rightsquigarrow \omega_1$. By Lemma 5.7, $\omega_1 \not\rightsquigarrow U_1$. Also by Lemma 5.7, $\omega_1 \not\rightsquigarrow U_i$ $(i = 2, \ldots, m)$. Similarly $\omega_2 \not\rightsquigarrow U_i$ $(i = 1, \ldots, m)$. So there exists $M \in \mathbf{Z}^+$ such that $\omega_1^M \not\twoheadrightarrow U_i$ $(i = 1, \ldots, m)$ and $\omega_2^M \not\twoheadrightarrow U_i$ $(i = 1, \ldots, m)$. Let $k \geq M$. By Theorem 4.8, there exists an indecomposable sequence $\langle V_j \rangle_{j=1}^p$ between ω_1 and ω_2 such that for each j, $\omega_1^k | V_j$ or $\omega_2^k | V_j$. So since $k \geq M$, $V_j \not\text{---} U_i$ for any $j = 1, \ldots, p$ and $i = 1, \ldots, m$. So $\langle U_i \rangle_{i=0}^{m+1}$, $\langle V_j \rangle_{j=1}^p$ describe an irreducible polygon with more than three vertices and containing ω_1, ω_2 and at least one point of length $> k$. Since k is any integer $\geq M$, we are done. $\square$

5.10. Corollary. *Let $\omega_1, \omega_2 \in \mathcal{C} = \mathcal{C}(A_1, \ldots, A_t)$, $t \geq 2$. If $\omega_1 \not\rightsquigarrow \omega_2$, then there exists an infinite number of irreducible polygons with more than three vertices and containing ω_1 and ω_2.*

Proof. If $\omega_1 \mathrel{\not-} \omega_2$, then the proof of Theorem 5.9 yields the result. So assume $\omega_1 \mathrel{-} \omega_2$. By Lemma 5.7, $\omega_2 \not\rightsquigarrow \omega_1$. So for some $i \in \mathbf{Z}^+$, $\omega_1^i \mathrel{\not-} \omega_2$, $\omega_1^i \mathrel{\not-} \omega_2^i$, $\omega_2^i \mathrel{\not-} \omega_1$. But now the result follows from the first half of the proof of Theorem 5.9. □

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